

Chapter 5. Electromagnetic Induction

Topics to cover:

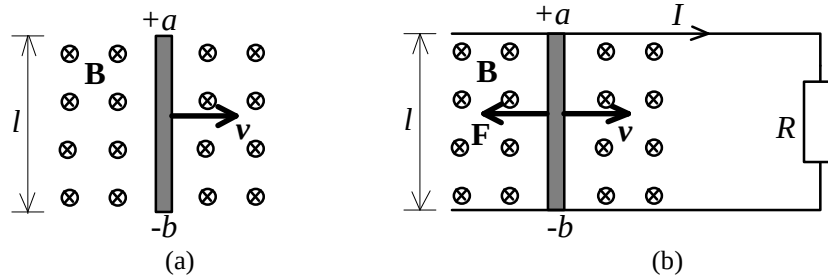
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| 1) <i>EMF in a Conductor Moving in a Static Magnetic Field</i> | 2) <i>Faraday's Law</i> |
| | 3) <i>Maxwell's Equations</i> |

The Motion of a Conductor through a Magnetic Field

Consider a conductor moving through a steady magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$ as shown below in diagram (a). The Lorentz force component $\mathbf{F}_m = Q(\mathbf{v} \times \mathbf{B})$ will cause free charges within the conductor to move; positive to the top, negative to the bottom. This generates an electric field $\mathbf{E}$ internal to the conductor. The force due to this field $\mathbf{F}_e = Q\mathbf{E}$ will oppose the free charge movement. Equilibrium results when $\mathbf{F}_m + \mathbf{F}_e = 0$, and the charges stop moving. An electric potential difference V exists between the top and the bottom of the conductor. This can be written as

$$V = -\mathbf{E} \cdot \mathbf{l} = -\frac{\mathbf{F}_e}{Q} \cdot \mathbf{l} = \frac{\mathbf{F}_m}{Q} \cdot \mathbf{l} = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{l}$$

where $\mathbf{l}$ is a vector pointing from $-b$ to $+a$ with magnitude l ; refer to the figure.



If this moving conductor was connected to an external circuit as shown in diagram (b) with total resistance R then a current I would be expected to flow, which can be calculated by the Ohm's law:

$$I = V/R = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{l} / R$$

Energy would be dissipated as heat at the rate of

$$P = I^2 R = \{(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{l}\}^2 / R \quad (\text{W})$$

When the current flows in the conductor, the force on the conductor carrying current I is

$$\mathbf{F} = Q(\mathbf{v}' \times \mathbf{B}) = It \left(\frac{1}{t} \mathbf{l} \times \mathbf{B} \right) = I \mathbf{l} \times \mathbf{B}$$

as $Q=It$ and $\mathbf{v}'=I/t$. (Note that $\mathbf{v}'$ is the velocity of the moving charges and not the velocity of the conductor). It can be seen that this force is in the direction opposite to the movement of the conductor.

We have seen that magnetic as well as electrostatic forces can act on charges within a conducting circuit and that it is the former which is directly responsible for producing a potential difference between terminals a and b . The work done in taking a charge from a to b via R is work done by electrostatic forces and is equal to the potential difference between a and b . This is not true for the return path from b to a along the conducting rod. Here the force exerted on a charge by the electrostatic field is balanced by the magnetic force resulting from the motion of the charge through the magnetic field. Thus no work is done when the charge is taken from b to a along the conducting rod. So, when a charge is taken from a to b via R and back to a along the conducting rod a net quantity of work is done, which equals the potential difference between a and b . It is no longer true, as in the purely electrostatic case to say that the total potential change around a closed path is zero. This could be expected in other situations where forces other than purely electrostatic forces act. (Consider a battery where chemical energy causes the flow of current).

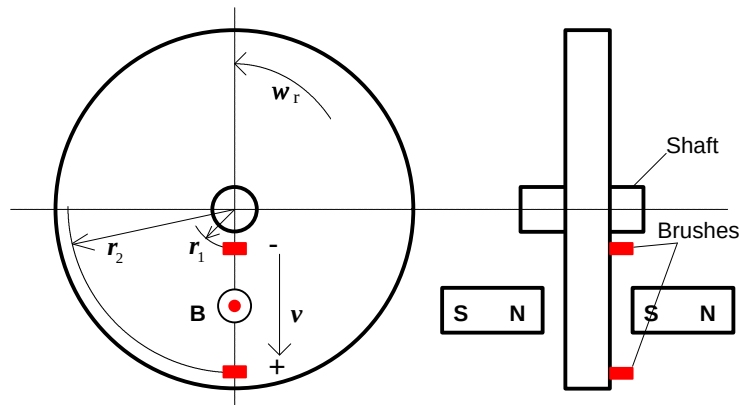
For a complete circuit including the conducting rod (or battery) it is useful to define an **electromotive force** (*emf*) e . The unit for e is volt (V). (Be careful: the symbol e is also used for the charge carried by an electron.) We can define e as the work done in moving a charge around the external circuit once. That is

$$e = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{l}$$

(In the previous figure, $\mathbf{l}$, $\mathbf{v}$ and $\mathbf{B}$ are all at right angles and thus $(\mathbf{v} \times \mathbf{B})$ is in the same direction as $\mathbf{l}$. Therefore $e = vBl$.)

Example:

Calculate the open circuit voltage between the brushes on a Faraday's disc as shown schematically in the diagram below.



Solution:

Choose a small line segment of length dr at position r ($r_1 \leq r \leq r_2$) from the center of the disc between the brushes. The induced emf in this elemental length is then

$$de = Bvdr = Bw_r r dr$$

where $v = rw_r$. Therefore,

$$e = \int de = \int_{r_1}^{r_2} Bw_r r dr = w_r B \left. \frac{r^2}{2} \right|_{r_1}^{r_2} = w_r B \frac{r_2^2 - r_1^2}{2}$$

Faraday's Law

When the magnetic flux linking a circuit changes, an emf is induced in the circuit. **Faraday's Law** states that the emf equals the rate of change of flux. Mathematically,

$$e = -\frac{df}{dt}$$

where the minus sign indicates the direction of the induced emf is such that any current produced by it tends to oppose the change in flux, which is known as the **Lenz's Law**.

More generally, when there are N circuits or N turns, the Faraday's law can be written as

$$e = -\frac{dl}{dt}$$

where $l = Nf$ is the total flux linkage.

Although the emf by Faraday's law is expressed in a form different from that induced in a moving rod, these two formulas are essentially the same. Consider the conducting rod moving in the magnetic field with velocity $\mathbf{v}$. Previously we defined

$$e = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{l}$$

and as previously stated, since $\mathbf{l}$, $\mathbf{v}$ and $\mathbf{B}$ are at right angles we can write

$$e = vlB$$

Noting that $lvdt$ is the area swept out by the rod in time interval dt and that $lvBdt$ is the decrease in the flux linked with the circuit during dt , we obtain

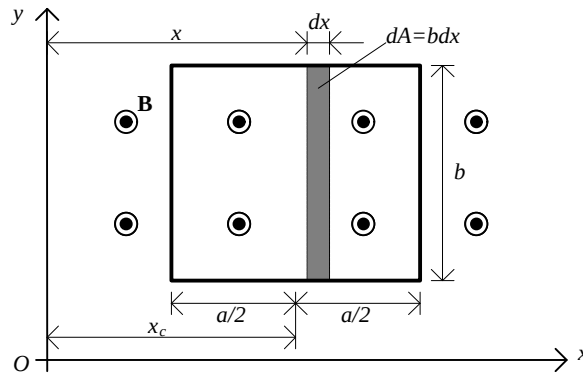
$$-df = lvBdt$$

and therefore

$$e = -\frac{df}{dt}$$

Example:

Consider a small rectangular coil of N turns wound on a non-magnetic former in a magnetic field of density $\mathbf{B}$ (perpendicular to the paper) as shown below.



(a) Calculate the flux linking the coil if $B = \hat{B} \sin ax$, where a is a constant.

Solution:

Here we consider a strip $dA = b dx$, and therefore $df = B dA$. The linking flux is then determined from

$$\begin{aligned}
 f &= \int_{x_c - \frac{a}{2}}^{x_c + \frac{a}{2}} \hat{B} \sin ax \cdot b dx \\
 &= -\frac{\hat{B}b}{a} [\cos ax]_{x_c - \frac{a}{2}}^{x_c + \frac{a}{2}} \\
 &= -\frac{\hat{B}b}{a} \left[\cos a \left(x_c + \frac{a}{2} \right) - \cos a \left(x_c - \frac{a}{2} \right) \right] \\
 &= \frac{2b}{a} \sin \left(\frac{aa}{2} \right) \hat{B} \sin ax_c \\
 &\quad \left(a \cos(u+v) - \cos(u-v) = -2 \sin u \sin v \right)
 \end{aligned}$$

and for N turns the flux linking the coil is

$$l = \frac{2Nb}{a} \sin \left(\frac{aa}{2} \right) \hat{B} \sin ax_c$$

(b) Calculate the emf induced in the coil when it moves in the x -direction with velocity $\mathbf{v}$.

Solution:

Consider

$$e = -\frac{dl}{dt} = -\frac{dl}{dx_c} \frac{dx_c}{dt}$$

and

$$v = \frac{dx_c}{dt}$$

Now

$$\begin{aligned} \frac{dl}{dx_c} &= \frac{2bN}{a} \sin\left(\frac{aa}{2}\right) \hat{B} \cos(ax_c) a \\ &= 2bN \sin\left(\frac{aa}{2}\right) \hat{B} \cos(ax_c) \end{aligned}$$

and therefore

$$e = -2bN \sin\left(\frac{aa}{2}\right) \hat{B} v \cos(ax_c)$$

(c) Determine the value of a which maximizes the emf determined in (b).

Solution:

From the above expression, we know the magnitude of emf is

$$\hat{E} = -2bN \sin\left(\frac{aa}{2}\right) \hat{B} v$$

When $\sin(aa/2) = \pm 1$ or $aa = (2n-1)\pi$, ($n=1,2,\dots$), it reaches the maximum, and therefore,

$$a = \pi/a \text{ (or an odd multiple)}$$

(d) Calculate the induced emf if $B = \hat{B} \sin(\omega t - ax)$ and the coil moves in the x -direction with velocity v , (ω constant). Hence determine when $v = \omega/a$.

Solution:

As $B = \hat{B} \sin(\omega t - ax)$, from (a) we have

$$\begin{aligned} l &= \frac{2bN}{a} \sin\left(\frac{aa}{2}\right) \hat{B} \sin(\omega t - ax_c) \\ &= \frac{2bN}{a} \sin\left(\frac{aa}{2}\right) \hat{B} \sin(\omega t - avt) \end{aligned}$$

and therefore

$$\begin{aligned} e &= -\frac{dl}{dt} \\ &= -\frac{2bN}{a} \sin\left(\frac{aa}{2}\right) \hat{B} \cos(\omega t - ax_c) (\omega - av) \end{aligned}$$

$$= -2bN \left(\frac{w}{a} - v \right) \sin \left(\frac{aa}{2} \right) \hat{B} \cos(wt - ax_c)$$

Thus, when $v = w/a$, the emf $e=0$.

EMF Expressed in Terms of Inductances

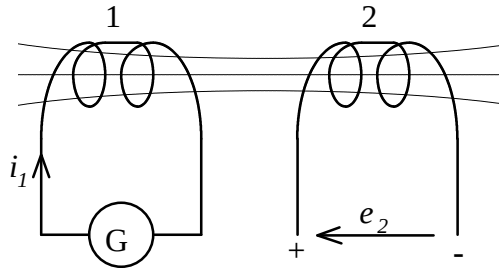
When a current I flows in a stationary circuit, a magnetic field will be established and the flux density at all points will be proportional to I . And so, for the flux ϕ linked with the circuit itself or the flux linkage l , we can write

$$l = LI$$

where the proportionality constant is the **self inductance** with the unit of henry (H). If the current in the circuit varies with time, an emf will be induced:

$$e = -\frac{dl}{dt} = -L \frac{dI}{dt}$$

with the **minus sign signifying that the emf tends to oppose the change in current**.



The situation depicted in the above figure represents a common situation where a generator (G) causes a current i_1 to flow in circuit 1 and a magnetic field is set up. Some of the magnetic flux link with circuit 2. At every point the flux density $\mathbf{B}$ will be proportional to i_1 and so the total flux linkage with circuit 2 will also be proportional to i_1 . Thus the flux linkage of circuit 2 due to the current in circuit 1, l_{21} , is

$$l_{21} = L_{21}i_1$$

where L_{21} is the **mutual inductance** between circuits 1 and 2.

It follows that the emf induced in circuit 2 can be written as

$$e_2 = -\frac{dl_{21}}{dt} = -L_{21} \frac{di_1}{dt}$$

Note here that the negative sign in this equation has no meaning unless we have a consistent convention for the positive directions of current and for the positive directions for winding the coils. In circumstances where the direction of the induced

emf is important it is a common practice to mark the corresponding ends of the windings by the dot convention.

Ideal Transformer

A transformer is an alternating current device that transforms voltages, currents and impedances. For ideal transformers, we assume that the permeability of the core approaches infinity, and that all power losses in the windings and the core are ignored. Thus, by the Ampere's law, we have

$$N_1 i_1 - N_2 i_2 = \frac{l}{\mu_0} f \Rightarrow 0$$

or

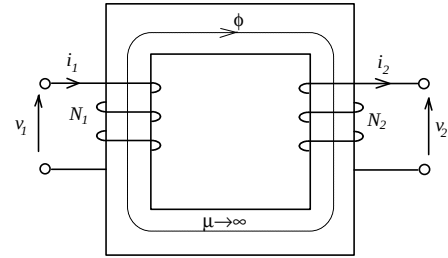
$$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$

Since the power losses are ignored, the input power must be equal to the output power:

$$v_1 i_1 = v_2 i_2$$

or

$$\frac{v_1}{v_2} = \frac{i_2}{i_1} = \frac{N_1}{N_2}$$



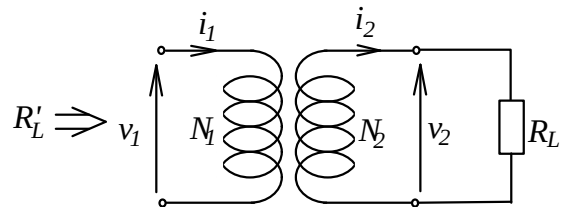
An ideal transformer

For resistance transform, we can write

$$R'_L = \frac{v_1}{i_1} = \left(\frac{N_1}{N_2} \right)^2 \frac{v_2}{i_2} = \left(\frac{N_1}{N_2} \right)^2 R_L$$

For impedance, we have

$$Z'_L = \left(\frac{N_1}{N_2} \right)^2 Z_L$$



Maxwell's Equations

The following table summarizes the governing equations (known as the Maxwell's equations) of electromagnetic fields, where the differential forms can be obtained by applying the **Stokes' Theorem**

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

where C is the contour bounding the surface S , and the **Gauss' theorem**

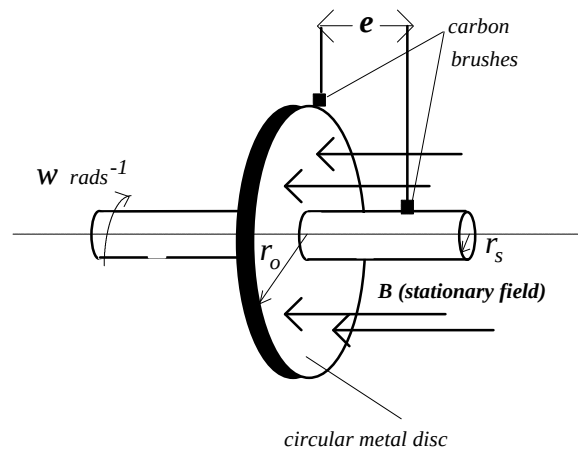
$$\int_V \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

where S is the closed surface bounding the volume V , to the corresponding integral forms of the Maxwell's equations.

Integral Form	Differential Form	Significance
$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{df}{dt}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
$\oint_C \mathbf{H} \cdot d\mathbf{l} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	Ampere's Law
$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	$\nabla \cdot \mathbf{D} = \rho$	Gauss' Law
$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	$\nabla \cdot \mathbf{B} = 0$	No isolated magnetic charge

Exercises

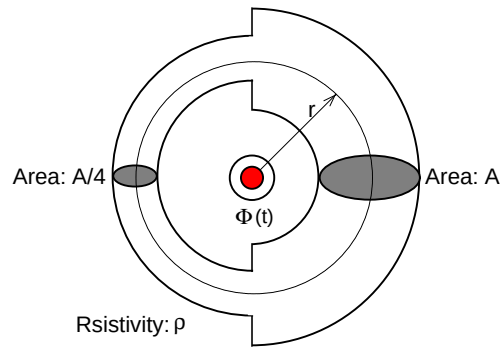
- The uniform magnetic field of density B tesla is perpendicular to the plane of the disc.



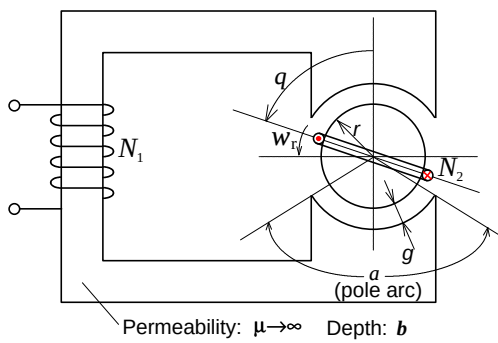
- Make a sketch showing the magnetic forces on the free electrons in the disc and the electrostatic forces on the free electrons.
- Derive an expression for the potential difference e between the carbon brushes on the outside rim and on the surface of the shaft.

(Answer: $\frac{1}{2} \omega B r_o^2$)

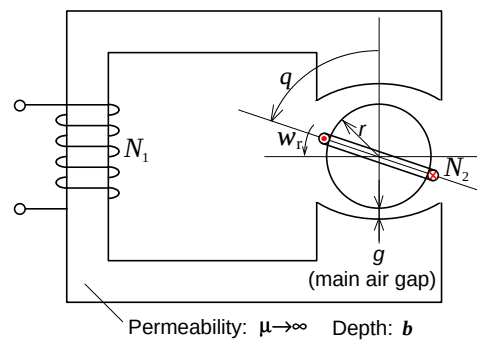
- Derive an expression for the current induced in the ring shown in the diagram below, neglecting the flux produced by the current itself.



3. Derive an expression for the *emf* in coil 2 in the diagram below and sketch the *emf* vs. time when
- (a) $i_1 = I_1$ a constant, and (b) $i_1 = I_m \sin \omega t$.
4. In the diagram below the poles are shaped to give a sinusoidal flux density $B = B_m \cos q$ at the rotor surface. Derive an expression for the *emf* in coil 2 and sketch the *emf* vs. time, when
- (a) $i_1 = I_1$ a constant, and (b) $i_1 = I_m \sin \omega t$.



(Problem 3)



(Problem 4)