

Introducing Faraday's Law

L. L. Tankersley and Eugene P. Mosca

Physics Department

United States Naval Academy

Annapolis, MD 21402

(410) 293-6653

tank@usna.edu

This note is to propose that several points be more heavily emphasized in introductory and intermediate presentations of Faraday's Law and that a commonly used notation be avoided. The use of careful and complete notation to ensure that the student is neither under-informed nor misled is most crucial. The Faraday **flux rule** relates the emf in a circuit to the rate of change of the magnetic flux through the circuit due both to the motion of the circuit and to the time variation of the magnetic field. In this paper, contributions to the net magnetic emf associated with the motion of conductors are referred to as motional emfs and those associated with the time dependence of the magnetic field are referred to as induced emfs (induction). For circuits in which a thin conductor defines a closed boundary, the Faraday (filamentary circuit) flux rule can be expressed as:

$$\mathcal{E}_{\text{magnetic}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left[\int_s \vec{B} \cdot \hat{n} da \right] \quad (1)$$

but not as:

$$\oint_c \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left[\int_s \vec{B} \cdot \hat{n} da \right] \quad (\text{misleading form}) \quad (2)$$

because Equation 2 can be interpreted to mean that the electric field circulates even for the case of a purely motional emf in a static magnetic field (see Equation 9a). However, the electric field circulates only in the case of induction. Note that we are restricting our attention to emfs associated with magnetic fields. Other sources, such as chemical cells, generate emfs by distinct means, and we refer you elsewhere for discussions of these subjects.^{1,2,3} The important point is that when considering *all* emfs, a circulating electric field exists only in the case of induction.

One of the first things that students learn in their introduction to electricity and magnetism is that for static situations the electric field does not circulate. As a consequence, an electrostatic potential can be defined such that the electric field is the negative of its gradient. The students must understand that this representation remains complete as long as and only as long as the magnetic field is static.

Some treatments introduce the Faraday flux rule and then analyze a motional emf device such as the slide wire generator in a static magnetic field to establish that the emf is equal to the (negative) time derivative of the magnetic flux through a surface bounded by the slidewire circuit. At this point, a simple statement that the circulation of the electric field remains zero so long as the magnetic field remains static is not sufficient if an expression equivalent to Equation 2 for the circulation of the electric field appears later. The magnetic flux can be time-dependent even for cases in which the

magnetic field is static, and so the student may conclude (incorrectly) that Equation 2 predicts that the electric field can circulate even if the magnetic field is static everywhere.

The relation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3)$$

frequently called the differential form of **Faraday's law**, can be transformed into an integral form

$$\mathcal{E}_{\text{induction}} = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} da \quad (4)$$

using standard theorems. These integrals are to be computed with paths and surfaces defined at and all fields evaluated at a single instant in time as measured in an inertial frame.⁴ We note that Equations 3 and 4 only describe induction, and that expressions for motional emfs must be deduced separately using the Lorentz force law

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}] \quad (5)$$

This approach leads to an expression for motional emfs

$$\mathcal{E}_{\text{motional}} = \oint_C (\vec{v}_c \times \vec{B}) \cdot d\vec{\ell} \quad (6)$$

where $\vec{v}_c$ is the velocity of the boundary element $d\vec{\ell}$.⁵ (At this point it should be emphasized that, in spite of the form of this expression, the magnetic force does no work.⁶) In contrast to Equation 4, Equation 1 is a complete statement of The Faraday flux rule which includes both motional emfs and induction, although Equation 5 is necessary to compute motional emfs for problems in which the circuit does not consist of a thin wire. The correct physics is always given by Equations 3 and 5.⁷

Many students in intermediate electromagnetism courses need more instruction on the relation between the total time derivative of an integral and the partial time derivative of its integrand. As an example, we examine the content of Equation 1 using Leibnitz's rule for differentiating integrals. This rule expands the total time derivative of an integral as the sum of two contributions, one associated with the motion of the boundaries of the integration interval and the other with the partial time derivative of the integrand. The Leibnitz rule is usually expressed as⁸

$$\frac{d}{dt} \left[\int_{x_1}^{x_2} b(x, t) dx \right] = \left[b(x_2, t) \frac{dx_2}{dt} - b(x_1, t) \frac{dx_1}{dt} \right] + \int_{x_1}^{x_2} \frac{\partial b(x, t)}{\partial t} dx \quad (7)$$

For a divergence-free field $\vec{B}(\vec{r}, t)$, this rule can be extended to a more general form^{9,10}

$$\frac{d}{dt} \left[\int_s \vec{B}(\vec{r}, t) \cdot \hat{n} da \right] = \oint_c \left[\vec{B}(\vec{r}, t) \cdot \left(\frac{d\vec{r}_c}{dt} \times d\vec{\ell} \right) \right] + \int_s \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \cdot \hat{n} da \quad (8)$$

where we identify $d\vec{r}/dt$ as $\vec{v}_c$, the velocity of the segment $d\vec{\ell}$ of the path bounding the integration surface. The cross product represents the rate at which area is swept out by $d\vec{\ell}$. By multiplying through Equation 8 by negative one and by changing the order of the products in the path integral we obtain

$$\frac{d}{dt} \left[\int_s \vec{B}(\vec{r}, t) \cdot \hat{n} da \right] = \oint_c (\vec{v}_c \times \vec{B}) \cdot d\vec{\ell} + \int_s \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \cdot \hat{n} da \quad (9a)$$

Substituting in Equation 9a for the terms corresponding to the expressions in Equations 1, 4, and 6 we have

$$\mathcal{E}_{\text{magnetic}} = \mathcal{E}_{\text{motional}} + \mathcal{E}_{\text{induction}} \quad (9b)$$

In spite of the curious completeness of Equation 1 as revealed in Equation 9b, the generation of a motional emf and induction are "two different phenomena."¹¹ The phenomena are clearly related. This relation can be seen by comparing the measurements made by an inertial observer in the situation that a conducting loop moves with constant velocity and fixed orientation in a static, non-uniform magnetic field with those of an inertial observer moving with the loop. The link between motional emf in one frame and the induced emf in the other is revealed not by the mathematics of the Leibnitz rule, but rather by the Principle of Relativity. Introductory presentations should give the distinct natures of the two phenomena the same weight that is given to their unity. Motional emfs cannot, in general, be transformed into examples of induction as there is no requirement that every point on a circuit be at rest in a single inertial frame. Constructive examples should be presented that identify the fields responsible for the work done on charges for motional and inductive phenomena. One might contrast the operation of a slidewire generator and a betatron. For the slidewire generator, a quasi-static Hall effect electric field does work on the current carriers as they follow a non-closed path.^{12,13} In the case of a betatron, a circulating electric field does work on the orbiting electrons.

As with Equation 4, the desire for careful, complete notation suggests that Maxwell's fourth equation be expressed as:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \cdot \hat{n} da \quad (10)$$

This equation also relates the circulation integral of a field to the time variations in the flux of another field. Moving boundary contributions are not to be included in this relation. Grouping the fields on the right hand side makes explicit that the surface integrations for the two fields are over the same surface S bounded by the curve C . The Maxwell equations describing the circulations of the fields are written, correctly and unambiguously, without total time derivatives in Equations 4 and 10.

Following these conventions, the local conservation of charge law is expressed as:

$$\oint_{S=\partial V} \vec{J} \cdot \hat{n} da = \int_V \left[\frac{\partial \rho}{\partial t} \right] dV \quad (11)$$

In summary, we recommend avoiding inexact or ambiguous representations of Faraday's Law, such as Equation 2, that blur the distinction between magnetic emf (motional plus induced) and the circulation of the electric field. The electric field circulates only if there is a time-dependent magnetic field. The fact that the magnetic force never does work should be reinforced whenever the magnetic force on charges is used as the basis for deriving expressions for motional emfs. The total time derivative of the flux of a divergence-free field through a surface can differ from the flux of the partial time derivative of that field if the boundary of the surface is not fixed in space. In the evaluation of the total time derivative of the magnetic flux through a circuit, the terms associated with the motion of the boundary do have an interpretation as the motional contribution to the total magnetic emf. However, they are not to be included in the expression for the circulation of the electric field.

Endnotes and References:

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- ¹ F. Reif, "Generalized Ohm's law, potential difference, and voltage measurements," Am. J. Phys. **50**, 1048-1049 (1982).
- ² D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, 1989), 2nd ed., pp. 277; 3rd ed., p. 297.
- ³ E. M. Purcel, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), pp. 134-138
- ⁴ Some authors [for example, see ref 10] use a form similar to Equation 2 in which the electric field is to be evaluated point by point in the local comoving frame of the line element. This elegant approach reveals part of the beauty of electromagnetic field theory, but, for the introductory student, it does not replace the inertial frame statement of Equation 4.
- ⁵ See Ref. 2, pp 280-3
- ⁶ E. P. Mosca, "Magnetic Forces Doing Work?," Am. J. Phys. **42**, 295-297 (1974).
- ⁷ R. P. Feynman, R.B. Leighton, and M. Sands, Addison-Wesley, Reading, 1964), p. 17-3
- ⁸ M. Boas, *Mathematical Physics* (John Wiley, New York, 1966), p. 162
- ⁹ See Ref.2, pp. 281-2
- ¹⁰ J. D. Jackson, *Classical Electrodynamics* (John Wiley, New York, 1975) 2nd ed., p. 212; 3rd ed., p. 210.
- ¹¹ See Ref. 7, p. 17-2.
- ¹² See Ref. 6, p. 297
- ¹³ See Ref. 2, p. 280